ODE Background: Introduction to SIR Epidemic Model and Simulating ODE Models in MATLAB

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Aims of These Slides

You will learn:

- Setup of a simple epidemic model
- A little about how the model behaves (will see more detail on this in the practical during the workshop)
- How to simulate the model in MATLAB

I assume you know about functions and M-files in MATLAB. If not, see:

https://www.mathworks.com/help/matlab/matlab_prog/create-functions-in-files.html

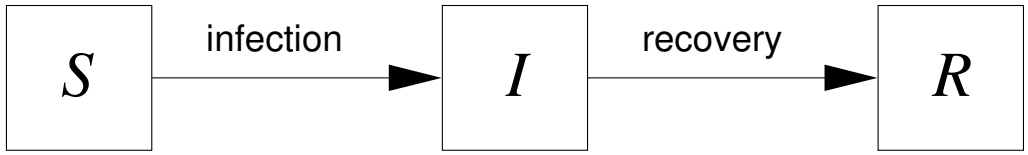
or

https://www.youtube.com/watch?v=qo3AtBoyBdM

SIR Model for Spread of Infection

Compartmental model, with three **state variables**:
Susceptibles, Infectives, Recovereds

Model tracks rate of change of these quantities



$$dS/dt = -\beta SI/N$$

$$dI/dt = \beta SI/N - \gamma I$$

$$dR/dt = \gamma I$$

Ignore births and deaths (e.g. short-lived outbreak)

"Standard incidence" term $\beta SI/N$ describes transmission process

β : "transmission parameter"
"well-mixed" population

Assume constant per-capita recovery rate of γ

$1/\gamma$ is average duration of infectiousness

Note: *S* + *I* + *R* = *N* (constant), so need only worry about *S* and *I*

Behavior of SIR Model

It turns out that model's behavior is governed by the value of the ratio $R_0 = \beta/\gamma$

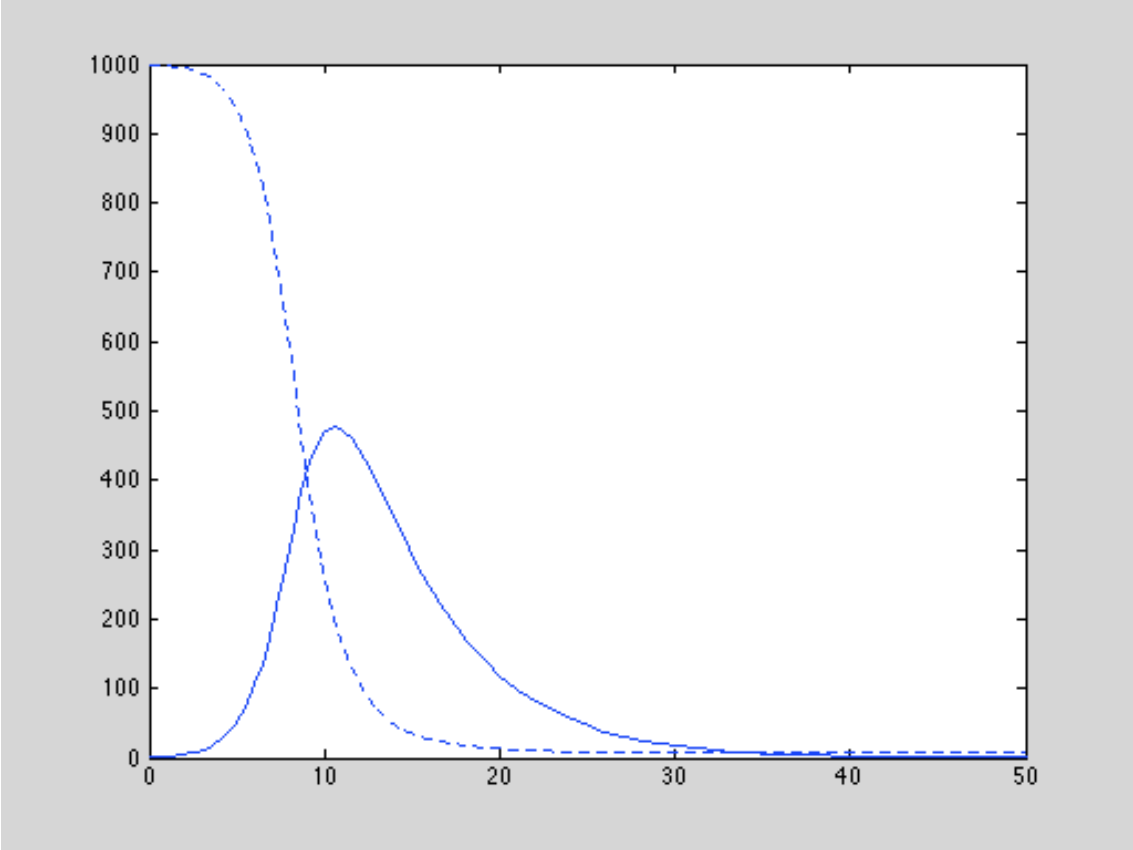
Outbreak can occur if $R_0 > 1$, cannot occur if $R_0 < 1$

$R_0 > 1$ plot:

$\beta = 1$, $\gamma = 0.2$, $N = 1000$
$S(0) = 999$, $I(0) = 1$

$S(t)$: dashed line

$I(t)$: solid line



Behavior of SIR Model

Behavior is governed by the value of the ratio $R_0 = \beta/\gamma$

Outbreak can occur if $R_0 > 1$, cannot occur if $R_0 < 1$

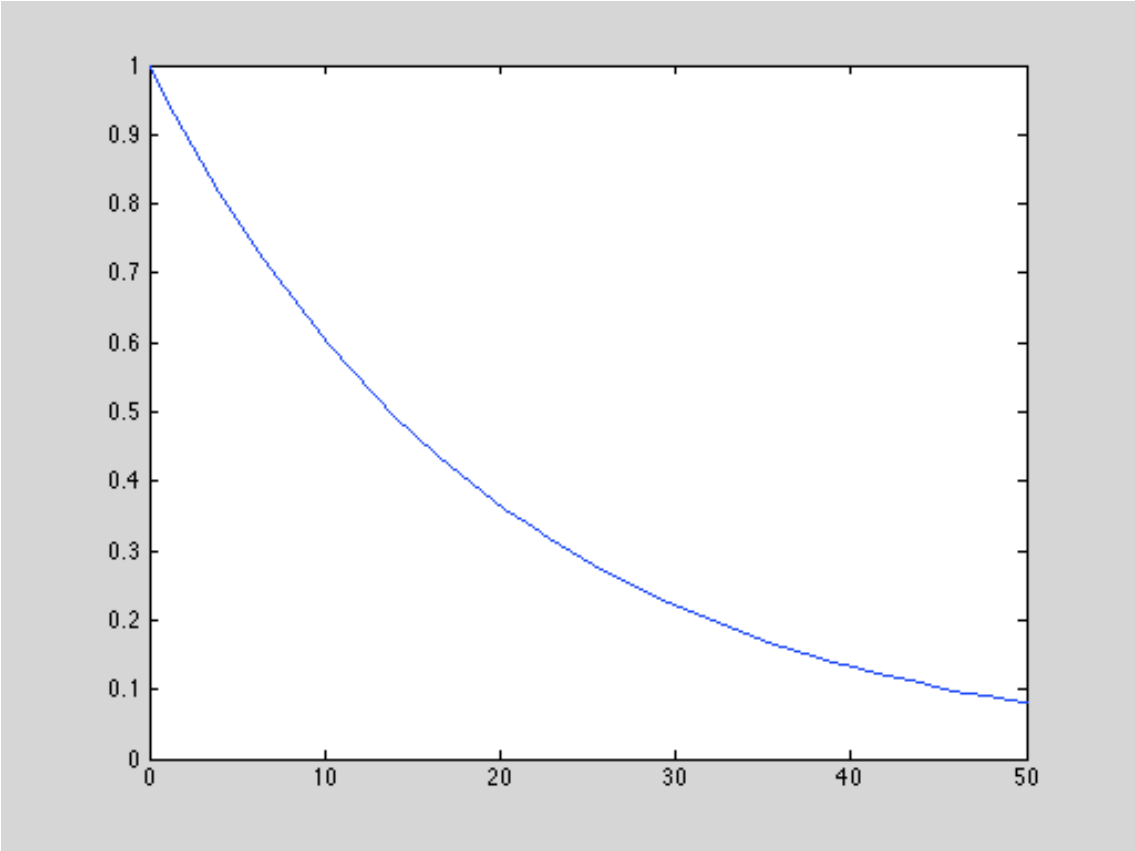
$R_0 < 1$ plot:

$\beta = 0.15$, $\gamma = 0.2$, $N = 1000$
$S(0) = 999$, $I(0) = 1$

$S(t)$: not shown (remains close to 999)

$I(t)$: solid line

note different scale on vertical axis



SIR Model : Forward Simulation

Nonlinearity of the transmission term means we cannot find an analytic solution of the model for *S* and *I* in terms of time

Numerically integrate (simulate) model in MATLAB, given a set of parameters and initial values for *S* and *I*

We shall use the `ode45` routine in MATLAB (syntax explained in two slides' time) to integrate the general initial value problem

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(t, \mathbf{y}) \quad ; \quad \mathbf{y}(0) = \mathbf{y}_0$$

Here **y** is a vector of states of the system (i.e. *S* and *I* for our SIR model), **f** is a vector function giving the time derivatives of the states, and $\mathbf{y}_0$ is a vector of initial conditions

SIR Model : Forward Simulation

We shall use the `ode45` routine in MATLAB. Syntax explained on the next slide

We need to tell ode45:

- the name of the function that calculates the values of the time derivatives of state variables (i.e. the components of the right hand sides of the differential equation)
- the time interval over which to integrate, and
- the initial conditions

Optionally, we can pass options that `ode45` is to use (e.g. integration tolerances) and a vector of parameters

MATLAB works with state variables arranged in a vector, so we shall use the first element (e.g. `y(1)`) to denote *S* and the second (e.g. `y(2)`) to denote *I*

ode45

```
[t,y]=ode45(@odefun,tspan,y0,options,pars);
```

`odefun` the name of the function that gives the right sides of our differential equations
(replace "odefun" with something more descriptive, but keep "@")

`tspan` vector that specifies the interval of times over which to integrate:

```
tspan = [t_initial, t_final]
```

or a vector of times at which we wish to obtain output :

```
tspan = [t_initial, t1, t2, … , t_final]
```

`y0` column vector of initial states (i.e. initial conditions) : `y0 = [ S0 ; I0 ]`

`options` options for the ODE solver, e.g. solution tolerances
use `[ ]` for no options; see `odeset` for information on options

`pars` a vector of parameter values that gets passed to `odefun`

`t` (returned) column vector of times at which output is given

`y` (returned) matrix of numerically calculated values of state variables over time

each row refers to a different time point, each column to a different state variable
e.g. `y(1,:)` are initial states, `y(end,:)` final states,
`y(:,2)` is a column vector of *I* values at all times — this is what we want to make an *I*(*t*) vs *t* plot

odefun

```
function f = odefun(t,y,pars)
```

Function `odefun` returns the entries of the right sides of the differential equations, $f(t,\mathbf{y})$, as a column vector

`t` (scalar) value of time at which to evaluate *f*

`y` column vector containing values of state variables

`pars` a vector of parameter values that gets passed to `odefun`

```
function f = sir_rhs(t,y,pars)

    f=zeros(2,1);

    beta=pars(1);
    gamma=pars(2);
    N=pars(3);

    S=y(1);
    I=y(2);

    f(1)=-beta*S*I/N;
    f(2)=beta*S*I/N-gamma*I;
end
```

need to return a column vector

could eliminate a number of these lines if we worked with y(1), pars(1) etc in the f(1) and f(2) lines

SIR Model Simulation

```
function sir_simulation

    beta=1.0;
    gamma=1.0/5.0;    % five day infectious period
    N=1000.0;

    pars=[beta,gamma,N];

    tspan=[0,50];      % simulate for 50 days

    y0=[999;1];        % one initial infective

    [t,y]=ode45(@sir_rhs,tspan,y0,[],pars);

    plot(t,y(:,2));   % plot prevalence of infection over time
end

function f = sir_rhs(t,y,pars)
    f=zeros(2,1);
    f(1)=-pars(1)*y(1)*y(2)/pars(3);
    f(2)=pars(1)*y(1)*y(2)/pars(3)-pars(2)*y(2);
end
```